Analytic and Algebraic Geometry

edited by
Tadeusz Krasiński
Stanisław Spodzieja

FACULTY OF MATHEMATICS
AND COMPUTER SCIENCE
UNIVERSITY OF ŁÓDŹ

Łódź 2013

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ŁÓDZKIEGO

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Preface

Annual Conferences in Analytic and Algebraic Geometry have been organized by Faculty of Mathematics and Computer Science of the University of Łódź since 1980. Until now, proceedings of these conferences (mainly in Polish) have comprised educational materials describing current state of a branch of mathematics, new approaches to known topics, and new proofs of known results (see the Internet page: http://konfrogi.math.uni.lodz.pl/).

The subject of the present volume include new results and survey articles concerning real and complex algebraic geometry, singularities of curves and hypersurfaces, invariants of singularities (the Milnor number, degree of $\mathcal{C}^0$-sufficiency), algebraic theory of derivations and others topics.

One remarkable element of this collection is an English translation of the Polish version, published in proceedings of the above mentioned conferences, of an article by Stanisław Łojasiewicz (1926-2002) devoted to the famous Hironaka theorem on resolution of singularities. It contains his original approach to the problem in the case of curves and coherent analytic sheaves on 2-dimensional manifolds. This interesting article has not yet been available in English. Additionally, we add a photo portrait of him and the facsimile of one page of his original handwritten manuscript.

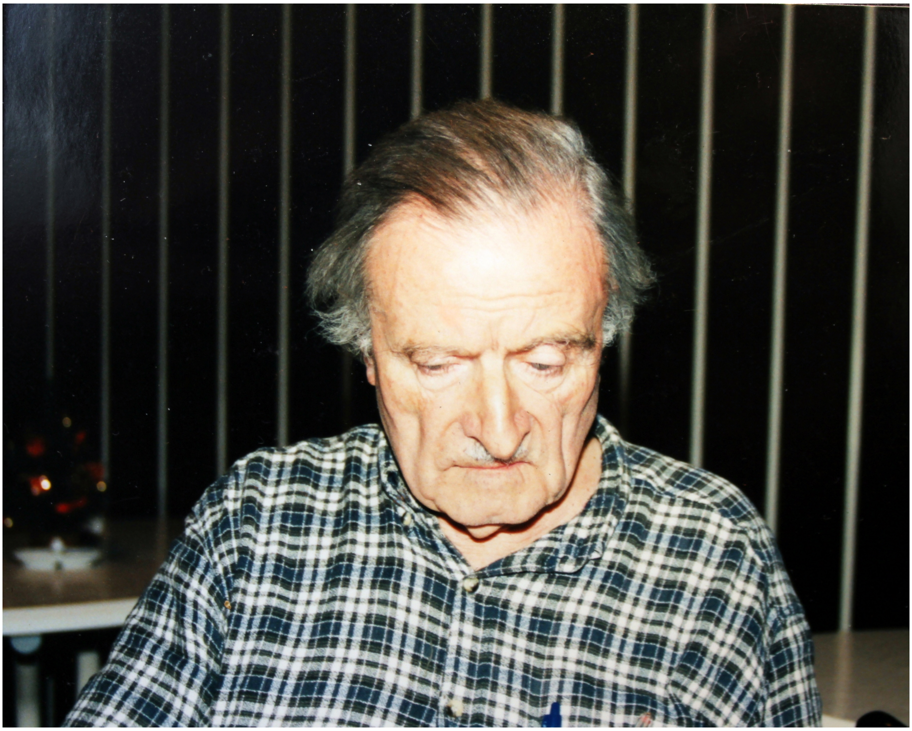
We would like to thank Arkadiusz Płoski for the help in preparing the volume, Michał Jankowski for designing the cover, referees for preparing reports of the articles and all participants of the Conferences for their good humor and enthusiasm in doing mathematics.

Finally, we would like to thank Stanisław Łojasiewicz jr and Anna Ostoja-Łojasiewicz, the heirs of Stanisław Łojasiewicz, for having agreed to include his article into this volume.

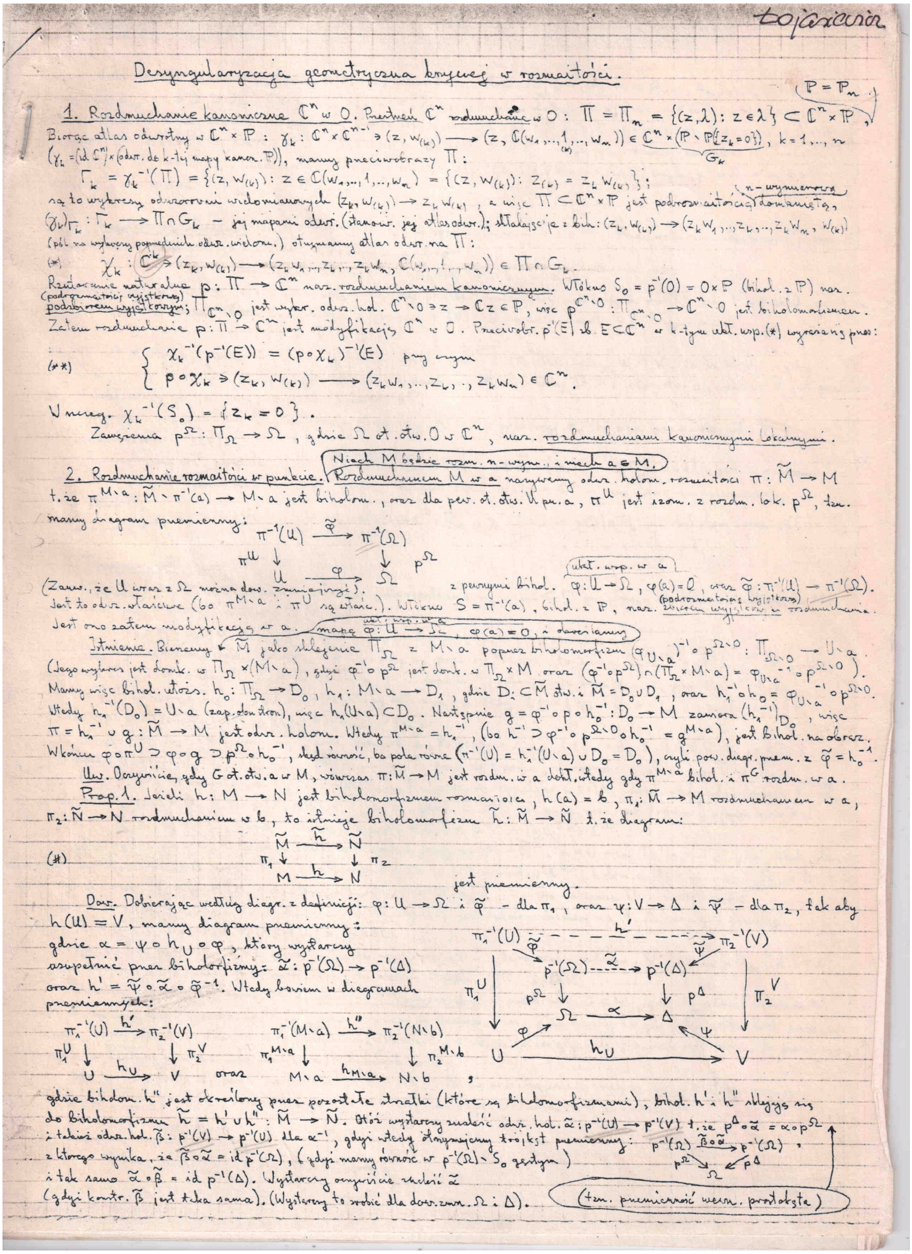
We dedicate the whole volume to the memory of Stanisław Łojasiewicz.

Tadeusz Krasiński
Stanisław Spodzieja

November 2013, Łódź



Stanisław Łojasiewicz (9 X 1926 – 14 XI 2002)
(The photo was taken by Przemysław Skibiński in 2000)


Łojasiewicz

Desyngularyzacja geometryczna krzywej w rozmaitości.

$(\mathbb{P} = \mathbb{P}_n)$

1. Rozdmuchanie kanoniczne $\mathbb{C}^n$ w 0. Przestrzeń $\mathbb{C}^n$ rozdmuchaną w 0: $\Pi = \Pi_n = \{(z,\lambda): z \in \lambda\} \subset \mathbb{C}^n \times \mathbb{P}$,
Biorąc atlas odwrotny w $\mathbb{C}^n \times \mathbb{P}$: $\gamma_k : \mathbb{C}^n \times \mathbb{C}^{n-1} \ni (z, w_{(k)}) \longrightarrow (z, \mathbb{C}(w_1,\dots,\underset{(k)}{1},\dots,w_n)) \in \mathbb{C}^n \times (\mathbb{P} \setminus \mathbb{P}(\{z_k=0\}))$, $k=1,\dots,n$
($\gamma_k = (\mathrm{id}\,\mathbb{C}^n) \times$ (odwr. do k-tej mapy kanon. $\mathbb{P}$)), mamy przeciwobrazy Π:
$\Gamma_k = \gamma_k^{-1}(\Pi) = \{(z, w_{(k)}): z \in \mathbb{C}(w_1,\dots,1,\dots,w_n)\} = \{(z, w_{(k)}): z_{(k)} = z_k w_{(k)}\}$;
są to wykresy odwzorowań wielomianowych $(z_k, w_{(k)}) \to z_k w_{(k)}$, a więc $\Pi \subset \mathbb{C}^n \times \mathbb{P}$ jest podrozmaitością (n-wymiarową) domkniętą,
$(\gamma_k)_{\Gamma_k} : \Gamma_k \longrightarrow \Pi \cap G_k$ – jej mapami odwr. (stanowi. jej atlas odwr.); składając je z bih.: $(z_k, w_{(k)}) \to (z_k w_1,\dots,z_k,\dots,z_k w_n, w_{(k)})$
(pól na wykresy poprzednich odwz. wielom.) otrzymamy atlas odwr. na Π:
(*) $\chi_k : \mathbb{C}^k \ni (z_k, w_{(k)}) \longrightarrow (z_k w_1,\dots,z_k,\dots,z_k w_n, \mathbb{C}(w_1,\dots,1,\dots,w_n)) \in \Pi \cap G_k$.
Rzutowanie naturalne $p : \Pi \to \mathbb{C}^n$ naz. rozdmuchaniem kanonicznym. Włókno $S_0 = p^{-1}(0) = 0 \times \mathbb{P}$ (bihol. z $\mathbb{P}$) naz. podzbiorem wyjątkowym (podrozmaitością wyjątkową); $\Pi_{\mathbb{C}^n \setminus 0}$ jest wykr. odwz. hol. $\mathbb{C}^n \setminus 0 \ni z \to \mathbb{C}z \in \mathbb{P}$, więc $p^{\mathbb{C}^n \setminus 0} : \Pi_{\mathbb{C}^n \setminus 0} \to \mathbb{C}^n \setminus 0$ jest biholomorfizmem.
Zatem rozdmuchanie $p : \Pi \to \mathbb{C}^n$ jest modyfikacją $\mathbb{C}^n$ w 0. Przeciwobr. $p^{-1}(E)$ zb. $E \subset \mathbb{C}^n$ w k-tym ukł. wsp. (*) wyraża się przez:

$$(**)\quad \begin{cases} \chi_k^{-1}(p^{-1}(E)) = (p \circ \chi_k)^{-1}(E) & \text{przy czym} \\ p \circ \chi_k \ni (z_k, w_{(k)}) \longrightarrow (z_k w_1,\dots,z_k,\dots,z_k w_n) \in \mathbb{C}^n. \end{cases}$$

W szczeg. $\chi_k^{-1}(S_0) = \{z_k = 0\}$.
Zawężenia $p^{\Omega} : \Pi_{\Omega} \to \Omega$, gdzie Ω ot. otw. 0 w $\mathbb{C}^n$, naz. rozdmuchaniami kanonicznymi lokalnymi.

2. Rozdmuchanie rozmaitości w punkcie. (Niech M będzie rozm. n-wym., i niech $a \in M$.) Rozdmuchaniem M w a nazywamy odwz. holom. rozmaitości $\pi : \tilde{M} \to M$ t. że $\pi^{M \setminus a} : \tilde{M} \setminus \pi^{-1}(a) \to M \setminus a$ jest bihol., oraz dla pew. ot. otw. U pu. a, π^U jest izom. z rozdm. lok. p^{Ω}, tzn.

The facsimile of the first page of the Polish handwritten version of the article (1988) by Stanisław Łojasiewicz, translated in this volume.

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GEOMETRIC DESINGULARIZATION OF CURVES IN MANIFOLDS *) **)

STANISŁAW ŁOJASIEWICZ

1. Introduction

The article does not pretend to any originality. In the literature there exists a number of descriptions of desingularizations in the case of curves. Deciding for this description the author think it is worth looking in details into this fascinating topic in an easily accessible case, namely – in the effects of multi blowings-up for curves in manifolds and for coherent sheaves on 2-dimensional manifolds.

All the needed facts from analytic geometry can be find in the author's books [L1], [L2].

2. The canonical blowing-up of $\mathbb{C}^n$ at 0

The *blow-up of* $\mathbb{C}^n$ *at* 0 is

$$\Pi = \Pi_n = \{(z, \lambda) : z \in \lambda\} \subset \mathbb{C}^n \times \mathbb{P}, \quad \mathbb{P} = \mathbb{P}_{n-1}.$$

Taking the inverse atlas for $\mathbb{C}^n \times \mathbb{P}$

$$\gamma_k : \mathbb{C}^n \times \mathbb{C}^{n-1} \ni (z, w_{(k)}) \mapsto (z, \mathbb{C}(w_1, ..., \underset{(k)}{1}, ..., w_n)) \in \mathbb{C}^n \times \{\mathbb{P} \setminus \mathbb{P}(\{z_k = 0\})\} = G_k, \; k = 1, ..., n,$$

2010 *Mathematics Subject Classification.* Primary 32Sxx, Secondary 14Hxx.
Key words and phrases. Resolution of singularities, curve, blowing-up, coherent analytic sheaf.

*) This article was published (in Polish) in the proceedings of Xth Workshop on Theory of Extremal Problems (1989) and has never appeared in translation elsewhere. To honor this outstanding mathematician (who passed away in 2002) this article was translated into English (by T. Krasiński) in order to make it accesible to the mathematical community.

**) The translator thanks Dinko Pervan (an Erasmus student from Croatia) for preparing the article in TeX and W. Kucharz, A. Płoski and Sz. Brzostowski for improving the English text.

(that is $\gamma_k = (\mathrm{id}\,\mathbb{C}^n) \times$ (inverse mapping to the k-th canonical map on $\mathbb{P}$)), we have the inverse images of Π

$$\Gamma_k = \gamma_k^{-1}(\Pi) = \{(z, w_{(k)}) : z \in \mathbb{C}(w_1, ..., 1, ..., w_n)\} = \left\{(z, w_{(k)}) : z_{(k)} = z_k w_{(k)}\right\};$$

they are graphs of the polynomial mappings $(z_k, w_{(k)}) \to z_k w_{(k)}$, whence $\Pi \subset \mathbb{C}^n \times \mathbb{P}$ is an n-dimensional closed submanifold, $(\gamma_k)_{\Gamma_k} : \Gamma_k \to \Pi \cap G_k$ – its inverse maps (they give an inverse atlas on Π); composing them with biholomorphisms: $(z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n, w_{(k)})$ (domains onto the graphs of the preceding polynomial mappings) we obtain an inverse atlas on Π

$$(*) \quad \chi_k : \mathbb{C}^n \ni (z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n, \mathbb{C}(w_1, ..., 1, ..., w_n)) \in \Pi \cap G_k.$$

The canonical projection $p : \Pi \to \mathbb{C}^n$ is called the canonical blowing-up. The fiber $S_0 = p^{-1}(0) = 0 \times \mathbb{P}$ (biholomorphic to $\mathbb{P}$) is called the exceptional set (the exceptional submanifold); $\Pi_{\mathbb{C}^n \setminus 0}$ is the graph of the holomorphic mapping $\mathbb{C}^n \setminus 0 \ni z \to \mathbb{C}z \in \mathbb{P}$, whence $p^{\mathbb{C}^n \setminus 0} : \Pi_{\mathbb{C}^n \setminus 0} \to \mathbb{C}^n \setminus 0$ is a biholomorphism. Hence the blowing-up $p : \Pi \to \mathbb{C}^n$ is a modification of $\mathbb{C}^n$ at 0. The inverse image $p^{-1}(E)$ of a set $E \subset \mathbb{C}^n$ in the k-th coordinate system $(*)$ can be expressed by

$$(**) \qquad \begin{cases} \chi_k^{-1}(p^{-1}(E)) = (p \circ \chi_k)^{-1}(E) \text{ where} \\ p \circ \chi_k \ni (z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n) \in \mathbb{C}^n. \end{cases}$$

In particular $\chi_k^{-1}(S_0) = \{z_k = 0\}$.

The restrictions $p^{\Omega} : \Pi_{\Omega} \to \Omega$, where Ω is an open neighbourhood of 0 at $\mathbb{C}^n$, are called the local canonical blowings-up.

3. The blowing-up of a manifold at a point

Let M be an n-dimensional manifold and $a \in M$. A blowing-up of M at the point a is a holomorphic mapping of manifolds $\pi : \bar{M} \to M$ such that $\pi^{M \setminus a} : \bar{M} \setminus \pi^{-1}(a) \to M \setminus a$ is a biholomorphism and for an open neighbourhood U of a, the mapping π^U is isomorphic to a local canonical blowing-up p^{Ω} i.e. we have a commutative diagram

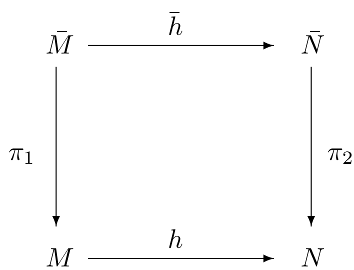
$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\bar{\phi}} & p^{-1}(\Omega) \\ \downarrow \pi^U & & \downarrow p^{\Omega} \\ U & \xrightarrow{\phi} & \Omega \end{array}$$

for some biholomorphisms $\phi : U \to \Omega$, $\phi(a) = 0$ and $\bar{\phi} : \pi^{-1}(U) \to p^{-1}(\Omega)$. (Notice that U and Ω can be abitrarily diminished). π is a proper mapping (because $\pi^{M \setminus a}$ and π^U are proper). The fiber $S = \pi^{-1}(a)$, biholomorphic to $\mathbb{P}$, is called the exceptional set (the exceptional submanifold) of the blowing-up. Thus π is a modification of M at a.

The existence of blowing-up. We take a chart (a coordinate system) at a: $\phi : U \to \Omega$, $\phi(a) = 0$, and define $\bar{M}$ as a gluing-up of π_Ω with $M \backslash a$ by the biholomorphism $(\phi_{U\backslash a})^{-1} \circ p^{\Omega\backslash 0} : \Pi_{\Omega\backslash 0} \to U\backslash a$. (Its graph is closed in $\Pi_\Omega \times (M\backslash a)$ because $\phi^{-1} \circ p^\Omega$ is a closed set in $\Pi_\Omega \times M$ and $\left(\phi^{-1} \circ p^\Omega\right) \cap (\Pi_\Omega \times M\backslash a) = \phi^{-1}_{U\backslash a} \circ p^{\Omega\backslash 0})$. So we have the identifying biholomorphisms $h_0 : \Pi_\Omega \to D_0$, $h_1 : M\backslash a \to D_1$, where $D_i \subset \bar{M}$, $i = 0, 1$, are open sets, $\bar{M} = D_0 \cup D_1$ and $h_1^{-1} \circ h_0 = \phi^{-1}_{U\backslash a} \circ p^{\Omega\backslash 0}$. Hence $h_1^{-1}(D_0) = U\backslash a$ (the domains of both sides) which implies $h_1(U\backslash a) \subset D_0$. Next $g = \phi^{-1} \circ p \circ h_0^{-1} : D_0 \to M$ contains $(h_1^{-1})_{D_0}$, and hence $\pi = h_1^{-1} \cup g : \bar{M} \to M$ is a holomorphic maping. Then $\pi^{M\backslash a} = h_1^{-1}$ (because $h^{-1} \supset \phi^{-1} \circ p^{\Omega\backslash 0} \circ h_0^{-1} = g^{M\backslash a}$) is a biholomorphism on the image. At last, $\phi \circ \pi^U \supset \phi \circ g \supset p^\Omega \circ h_0^{-1}$ which implies the equality, because the domains are equal ($\pi^{-1}(U) = h_1^{-1}(U\backslash a) \cup D_0 = D_0$), whence the above diagram is commutative with $\bar{\phi} := h_0^{-1}$.

Remark 1. *Obviously, if G is an open neighbourhood of a at M then $\pi : \bar{M} \to M$ is a blowing-up at a if and only if $\pi^{M\backslash a}$ is a biholomorphism and π^G is a blowing-up at a.*

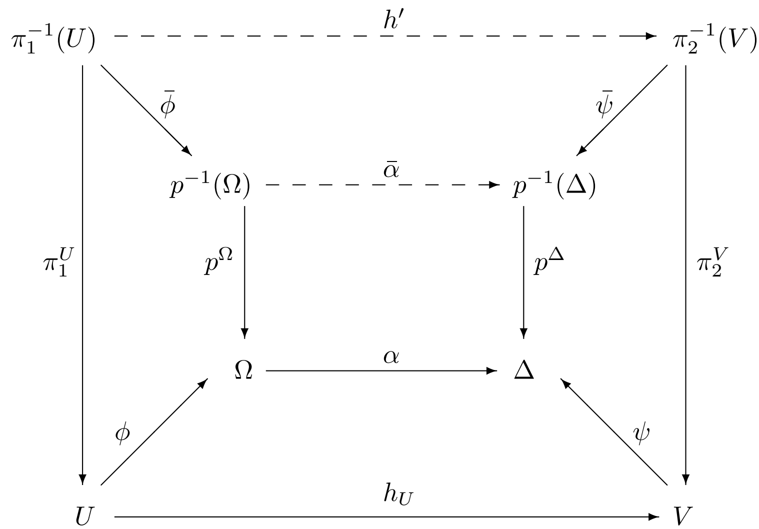
Proposition 1. *If $h : M \to N$ is a biholomorphism of manifolds, $h(a) = b$, $\pi_1 : \bar{M} \to M$ is a blowing-up at a, $\pi_2 : \bar{N} \to N$ a blowing-up at b, then there exists a biholomorphism $\bar{h} : \bar{M} \to \bar{N}$ such that the diagram*



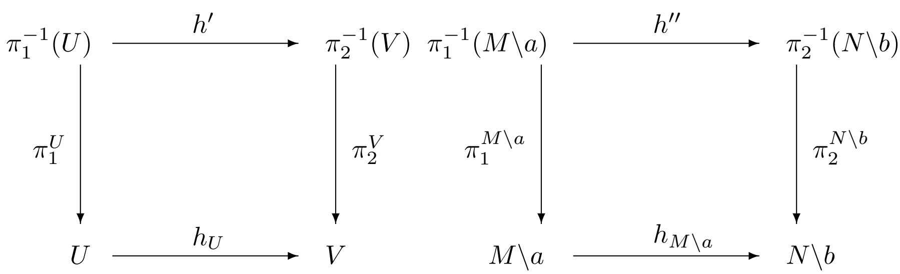
(#)

is commutative

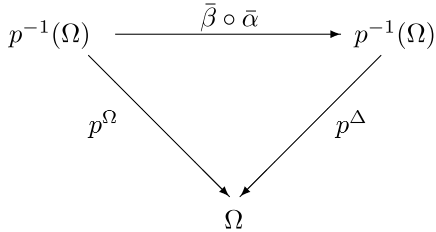
Dowód. Choosing by definition: $\phi : U \to \Omega$ and $\bar{\phi}$ - for π_1, and $\psi : V \to \Delta$ and $\bar{\psi}$ - for π_2, such that $h(U) = V$, we have a commutative diagram



where $\alpha := \psi \circ h_U \circ \phi^{-1}$, and it suffices to complement it by biholomorphisms: $\bar{\alpha} : p^{-1}(\Omega) \to p^{-1}(\Delta)$ and $h' := \bar{\psi}^{-1} \circ \bar{\alpha} \circ \bar{\phi}$. Then in the commutative diagrams



where the biholomorphism h'' is defined by the remaining arrows (which are biholomorphisms), the biholomorphisms h' and h'' give rise to a biholomorphism $\bar{h} = h' \cup h'' : \bar{M} \to \bar{N}$. In fact, it suffices to find a holomorphic mapping $\bar{\alpha}$: $p^{-1}(\Omega) \to p^{-1}(\Delta)$ such that $p^\Delta \circ \bar{\alpha} = \alpha \circ p^\Omega$ (i.e. the commutativity of the inner rectangle) and a similar holomorphic mapping $\bar{\beta} : p^{-1}(\Delta) \to p^{-1}(\Omega)$ for α^{-1}, since then we obtain the commutative triangle



which implies $\bar{\beta}\circ\bar{\alpha} = \mathrm{id}_{p^{-1}(\Omega)}$ (because we have the equality on the dense set $p^{-1}(\Omega)\setminus S_0$), and similarly $\bar{\alpha}\circ\bar{\beta} = \mathrm{id}_{p^{-1}(\Delta)}$. Obviously it suffices to find $\bar{\alpha}$ (because the construction of $\bar{\beta}$ is analogous) for sufficiently small Ω and Δ.

According to the Hadamard Lemma (since $\alpha(0) = 0$) one can choose neighbourhoods Ω, Δ such that $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i(z) = \sum_{j=1}^{n} a_{ij}(z)z_j$ and $\det a_{ij}(z) \neq 0$ in Ω. Define $a(z, w) = (\sum_{j=1}^{n} a_{1j}(z)w_j, ..., \sum_{j=1}^{n} a_{nj}(z)w_j)$ in $\Omega\times\mathbb{C}^n$; then $a(z, z) = \alpha(z)$ and $a(z, w) \neq 0$ for $w \neq 0$. Hence we may define a holomorphic mapping $\bar{a} : \Omega\times\mathbb{P} \ni (z, \mathbb{C}w) \to (\alpha(z), \mathbb{C}a(z, w)) \in \Delta\times\mathbb{P}$. Since $\bar{a}(z, \mathbb{C}z) = (\alpha(z), \mathbb{C}\alpha(z))$ for $z \in \Omega\setminus 0$ and $\bar{a}(0\times\mathbb{P}) \subset 0\times\mathbb{P}$, then we have the holomorphic restriction $\bar{\alpha} = \bar{a}_{\Pi_\Omega} : \Pi_\Omega \to \Pi_\Delta$, and hence $p^\Delta(\bar{\alpha}(z, \mathbb{C}z)) = \alpha(z) = \alpha(p^\Omega(z, \mathbb{C}z))$ for $z \in \Omega\setminus 0$, that is $p^\Delta\circ\bar{\alpha} = \alpha\circ p^\Omega$ by density of $\Pi_{\Omega\setminus 0}$ in Π_Ω. □

4. The proper inverse image

Let $\pi : \bar{M} \to M$ be a blowing-up at a point $a \in M$. The proper inverse image (by π) of a set $V \subset M$ closed in a neighbourhood of a (i.e. $V\cap U$ is a closed set in U for some neighbourhood U of a) is defined by

$$\bar{V} = \text{the closure of the set } \pi^{-1}(V\setminus a) = \pi^{-1}(V)\setminus S \text{ in } \pi^{-1}(V).$$

(It is obtained from the set $\pi^{-1}(V)\setminus S$ by adding to it its accumulation points belonging to S). If V is analytic in a neighbourhood of a then $\bar{V}$ is analytic in a neighbourhood of the exceptional set S (since $\pi^{-1}(V)$ and S are analytic in a neighbourhood of S). Obviously

$$\pi^{-1}(V) = \bar{V}\cup S.$$

If U is an open neighbourhood of a, then the proper inverse image of the set $V\cap U$ is $\bar{V}\cap\pi^{-1}(U)$. If $W \subset V$ then $\bar{W} \subset \bar{V}$, and if $V = \bigcup_{i=1}^{k} Z_i$, then $\bar{V} = \bigcup_{i=1}^{k} \bar{Z}_i$, (provided W, Z_i are closed in a neighbourhood of a). If $D \supset V$ is an open neighbourhood of a then $\bar{V}$ is the proper inverse image of V if and only if it is the same by the blowing-up π^D.

In Proposition 1 the biholomorphism $\bar{h}$ sends the exceptional submanifold $\pi_1^{-1}(a)$ onto the exceptional submanifold $\pi_2^{-1}(b)$, and the proper inverse image of V onto the proper inverse image of $h(V)$.

The proper inverse image of a linear subspace $L \subset \mathbb{C}^n$ of dimension k by the canonical blowing-up is $\bar{L} = \{(z,\lambda) \in L \times \mathbb{P}(L) : z \in \lambda\}$; it is a submanifold of dimension k and $p_{\bar{L}} : \bar{L} \to L$ is a blowing-up at 0. (For taking an isomorphism $\chi : L \to \mathbb{C}^k$ we have the commutative diagram

$$\begin{array}{ccc} \bar{L} & \xrightarrow{\ \psi_{\bar{L}}\ } & \Pi_k \\ {\scriptstyle p_{\bar{L}}}\downarrow & & \downarrow{\scriptstyle p_k} \\ L & \xrightarrow{\ \chi\ } & \mathbb{C}^k \end{array}$$

where $\psi = \chi \times \chi' : L \times \mathbb{P}(L) \to \mathbb{C}^k \times \mathbb{P}_k$, $\chi' : \mathbb{P}(L) \ni \lambda \to \chi'(\lambda) \in \mathbb{P}_k$ are biholomorphisms and $\psi(\overline{L}) = \Pi_k$).

5. The transversality

Proposition 2. *If M is a linear space of dimension n then linear subspaces $L_1, ..., L_r \subset M$ intersect transversally (in M) if and only if in some linear coordinate system in M it is*

$$L_i = \{z_\upsilon = 0, \upsilon \in I_i\}, \quad \textit{where } I_1, ..., I_r \subset \{1, ..., n\} \textit{ are disjoint.}$$

Dowód. The sufficiency is obvious because $\operatorname{codim} L_i = \# I_i$. Conversely, if L_i intersect transversally, then the sum $\sum L_i^\perp = (\bigcap L_i)^\perp$ is direct because $\dim \sum L_i^\perp = \operatorname{codim} \bigcap L_i = \sum \operatorname{codim} L_i = \sum \dim L_i^\perp$. Hence there exists a basis $\phi_1, ..., \phi_n$ of the dual space M^* such that $\{\phi_\upsilon : \upsilon \in I_i\}$ generate $L_i^\perp$ where $I_i \subset \{1, ..., n\}$ are disjoint. Then $L_i = \{\phi_\upsilon = 0, \upsilon \in I_i\}$, that is $L_i = \{z_\upsilon = 0, \upsilon \in I_i\}$ in the coordinate system $\phi = (\phi_1, ..., \phi_n)$ (because $\phi^{-1}(\{z_\upsilon = 0, \upsilon \in I_i\}) = L_i$). □

Corollary 1. *If L_i, $i \in I$, intersect transversally and $J \subset I$, then also L_i, $i \in J$, intersect transversally. If $I \cap J = \emptyset$ and L_i, $i \in I \cup J$, intersect transversally then so do $\bigcap_I L_i$ and $\bigcap_J L_i$. If $L_1, ..., L_r, T$ intersect transversally then so do $L_1 \cap T, ..., L_r \cap T$ in T.*

Proposition 3. *If M is a manifold of dimension n, then submanifolds $N_1, ..., N_r$ intersect transversally at a point $a \in \bigcap N_i$ if and only if there exists a chart (a coordinate system at a) $\phi : U \to \Omega$, $\phi(a) = 0$, such that $\phi(N_i \cap U) = T_i \cap \Omega$, where*

$T_i \subset \mathbb{C}^n$ *are subspaces that intersect transversally, so it may be*

$$T_i = \{u_i = 0\}, \text{ where } z = (u_1, ..., u_r, v) \in \mathbb{C}^n = \mathbb{C}^{I_1} \times ... \times \mathbb{C}^{I_r} \times \mathbb{C}^J.$$

Dowód. The sufficiency is clear. For the necessity we may assume $M = \mathbb{C}^n$, $a = 0$ and $T_0 N_i = T_i$ as above. Then there exists an open neighbourhood $U = \Omega_1 \times ... \times \Omega_r \times \Delta$ of the origin in $\mathbb{C}^n$ and functions $\varepsilon_i(u_{(i)}, v)$ with values in $\mathbb{C}^{I_i}$, holomorphic in $U_i = \Omega_1 \times ..._{(i)}... \times \Omega_r \times \Delta$, such that $d_0 \varepsilon_i = 0$ and $N_i \cap U = \{u_i = \varepsilon_i(u_{(i)}, v), (u_{(i)}, v) \in U_i\}$. After shrinking U the mapping $\phi : U \ni z \to (u_1 - \varepsilon_1(u_{(1)}, v), ..., u_r - \varepsilon_r(u_{(r)}, v), v) \in \Omega$ is a biholomorphism onto a neighbourhood Ω of the origin and hence $N_i \cap U = \phi^{-1}(T_i)$ which implies $\phi(N_i \cap U) = T_i \cap \Omega$. □

Corollary 2. *If submanifolds N_i, $i \in I$, intersect transversally at a point a and $J \subset I$, then so do the submanifolds N_i, $i \in J$. If $I \cap J = \emptyset$ and submanifolds $N_i, i \in I \cup J$, intersect transversally at a then so do the submanifolds $\bigcap_I N_i$ and $\bigcap_J N_i$.*

Corollary 3. *If submanifolds N_i intersect transversally then $N = \bigcap N_i$ is a submanifold and* $\operatorname{codim} N = \sum \operatorname{codim} N_i$.

We say submanifolds N_i of a manifold M are mutually transversal in an open set $G \subset M$, if $N_i \cap G$ are closed and for each $a \in G$ submanifolds N_i containing a intersect transversally at a. Notice that if subspaces of a linear space intersect transversally then they are mutually transversal in this space (by Corollary 1 and from the fact that if subspaces intersect transversally, then they intersect transversally at each point of their intersection). Hence (by Proposition 3)

Corollary 4. *If submanifolds N_i intersect transversally at $a \in \bigcap N_i$, then they are mutually transversal in a neighbourhood of the point a.*

6. The effect of blowing-up

Let M be a manifold of dimension n and let $\pi : \bar{M} \to M$ be a blowing-up at point $a \in M$, and $S = \pi^{-1}(a) \subset \bar{M}$ – the exceptional set.

Proposition 4. *If $\Gamma \subset M$, $\Gamma \ni a$, is a submanifold of dimension s then its proper inverse image $\bar{\Gamma} \subset \bar{M}$ is a submanifold of dimension s which intersects S transversally and the submanifold $\bar{\Gamma} \cap S$ is biholomorphic to $\mathbb{P}_{s-1}$. Then $\pi_{\bar{\Gamma}} : \bar{\Gamma} \to \Gamma$ is a blowing-up at a with the exceptional set $\bar{\Gamma} \cap S$.*

Dowód. The set $\bar{\Gamma} \backslash S = \pi^{-1}(\Gamma \backslash a)$ is a submanifold of dimension s and $(\pi_{\bar{\Gamma}})^{\Gamma \backslash a} : \bar{\Gamma} \backslash S \to \Gamma \backslash a$ is a biholomorphism. Let us take a chart $\phi : U \to \Omega$, $\phi(a) = 0$, such that $\phi(\Gamma \cap U) = L \cap \Omega$, where $L = \{z_1 = ... = z_r = 0\}$ $(r = n - s)$. It suffices to show the proposition for π^U and $\Gamma \cap U$ because then the proper inverse image of $\Gamma \cap U$, that is $\bar{\Gamma} \cap \pi^{-1}(U)$, will be a submanifold (of dimension s) and $(\pi^U)_{\bar{\Gamma} \cap \pi^{-1}(U)} = (\pi_{\bar{\Gamma}})^{\Gamma \cap U}$ will be a blowing-up at a, whence $\bar{\Gamma}$ will be a submanifold and $\pi_{\bar{\Gamma}}$ a blowing-up at

a (see Remark 1). According to Proposition 1, it suffices to prove the proposition for p^Ω, $L\cap\Omega$ and 0. Since the proper inverse image of $L\cap\Omega$ is $\bar{L}\cap p^{-1}(\Omega)$, where $\bar{L}$ is the proper inverse image of L by p, and $(p^\Omega)_{\bar{L}\cap p^{-1}(\Omega)} = (p_{\bar{L}})^{L\cap\Omega}$, then it suffices to prove the proposition for p, L and 0. But $\bar{L}$ is a submanifold of dimension s, $p_{\bar{L}} : \bar{L} \to L$ is a blowing-up at 0 and $\bar{L}\cap S_0 = 0\times\mathbb{P}(L)$ (see Section 4). It remains to prove the transversality. We have (see $(**)$ in Section 2)

$$\chi_k^{-1}(p^{-1}(L)) = \begin{cases} \{z_k = 0\} \text{ if } \ k \leqslant r \\ \{z_k = 0\} \cup \{w_1 = ... = w_r = 0\} \text{ if } \ k > r, \end{cases}$$

so by $\chi_k^{-1}(S_0) = \{z_k = 0\}$ it is

$$\chi_k^{-1}(\bar{L}) = \begin{cases} \emptyset \text{ if } \ k \leqslant r \\ \{w_1 = ... = w_r = 0\} \text{ if } \ k > r, \end{cases}$$

whence (Proposition 2) the transversality of the intersection of $\bar{L}$ and S_0 follows. □

Proposition 5. *If submanifolds $\Gamma_1, ..., \Gamma_r \subset M$ intersect transversally at a and $\bar{\Gamma}_1, ..., \bar{\Gamma}_r$ are their proper inverse images then $\bar{\Gamma}_1, ..., \bar{\Gamma}_r, S$ are mutually transversal in a neighbourhood of S. If additionally Γ_i intersect transversally then the proper inverse image of $\Gamma = \bigcap \Gamma_i$ is $\bar{\Gamma} = \bigcap \bar{\Gamma}_i$.*

Dowód. If U is an open neighbourhood of a then the proper inverse image of $\Gamma_i\cap U$ $(\Gamma\cap U)$ is $\bar{\Gamma}_i\cap\pi^{-1}(U)$ $(\bar{\Gamma}\cap\pi^{-1}(U))$. By Propositions 3 and 1 it suffices to consider the canonical blowing-up p and $\Gamma_i = T_i = \{z_\upsilon = 0, \upsilon\in I_i\}$, I_i disjoint (by the fact $\bar{\Gamma}\backslash S = \bigcap(\bar{\Gamma}_i\backslash S)$). Let $\bar{T}_i$ denote the proper inverse image of T_i. We have (see $(**)$ in Section 2)

$$\chi_k^{-1}(p^{-1}(T_i)) = \begin{cases} \{z_k = 0\} \text{ if } \ k \in I_i \\ \{z_k = 0\} \cup \{w_\upsilon = 0, \upsilon \in I_i\} \text{ if } \ k \notin I_i, \end{cases}$$

so

$$\chi_k^{-1}(\bar{T}_i) = \begin{cases} \emptyset \text{ if } k \in I_i \\ \{w_\upsilon = 0, \upsilon \in I_i\} \text{ if } k \notin I_i, \end{cases}$$

which implies (Proposition 2) that $\bar{T}_i, ..., \bar{T}_r, S$ are mutually tranversal in Π. If $\bar{T}$ is the proper inverse image of $T = \bigcap T_i$ then $T = \{z_\upsilon = 0, \upsilon \in I\}$, where $I = \bigcup I_i$, and in the same way

$$\chi_k^{-1}(\bar{T}) = \begin{cases} \emptyset \text{ if } k \in I \\ \{w_\upsilon = 0, \upsilon \in I\} \text{ if } k \notin I, \end{cases}$$

so $\chi_k^{-1}(\bar{T}) = \bigcap \chi_k^{-1}(\bar{T}_i)$, whence $\bar{T} = \bigcap \bar{T}_i$. □

Let $\mathcal{C}(a) = \mathcal{C}(a, M)$ denote the set of curves $\Gamma \subset M$ (i.e. local analytic subsets of constant dimension 1) such that $a \in \Gamma$ and the germ Γ_a is irreducible. Then

(6.1) $$\mathcal{C}(a) = \bigcup_{p=1}^{\infty} \mathcal{C}_p(a),$$

where $\mathcal{C}_p(a) = \mathcal{C}_p(a, M)$ denotes the set of curves Γ in $\mathcal{C}(a)$ having, in some coordinate system ϕ in a (i.e. ϕ is a chart such that $\phi(a) = 0$), the form (that is $\phi(\Gamma)$ is a set of the form)

$$\text{(6.2)} \qquad \begin{cases} z_1 = t^p \\ v = c(t)t^q \end{cases} \qquad |t| < \sigma,$$

where $v = (z_2, ...z_n)$, $q \geqslant p$, and c is a holomorphic function in $\{|t| < \sigma\}$ $(\sigma > 0)$. (For it is of the form $\{f(t) : |t| < \sigma\}$, where f is a holomorphic mapping, a homeomorphism onto its image, $f(0) = 0$; it is $f(t) = g(t)t^p$, $p \geqslant 1$, $g(0) \neq 0$, and after changing the system of coordinates one may have $g_1(0) \neq 0$; then $g_1 = \gamma^p$ with γ holomorphic in a neighbourhood of the origin, $\gamma(0) \neq 0$, and it suffices to change the parameter putting $\tau = \gamma(t)t$ in a neighbourhood of the origin). In particular, $\mathcal{C}_1(a)$ is the set of all curves $\Gamma \ni a$ smooth at a.

A set Γ_0 of the form (6.2) (without any restriction on q) is always a curve in $\mathbb{C}^n$ having its germ irreducible at 0. (For the mapping $\{|t| < \sigma\} \ni t \to (t^p, c(t)t^q) \in \{|z_1| < \sigma^p\} \subset \mathbb{C}^n$ is proper). Let us notice that replacing σ by $0 < \bar{\sigma} < \sigma$ we obtain an open neighbourhood of 0 in Γ_0 (precisely $\Gamma_0 \cap \{|z_1| < \bar{\sigma}^p\}$). If $0 < q < p$ and $c(0) \neq 0$ then $\Gamma_0 \in \mathcal{C}_q$. In fact, if for example $c_2(0) \neq 0$ then (changing the parameter to $\tau = t\gamma(t)$, where $\gamma^q = c_2$) for sufficiently small ε, a neighbourhood U_ε of the origin and holomorphic b_i, the sets $\Gamma_\varepsilon = \{z_1 = t^p, v = c(t)t^q, t \in U_\varepsilon\} = \{z_2 = \tau^q, z_i = b_i(\tau)\tau^q,\ i \neq 2,\ |\tau| < \varepsilon\}$ are neighbourhoods of 0 in Γ_0. But $\Gamma_{\varepsilon_0} \subset \Gamma_0 \cap \{|z_1| < \sigma_0\} \subset \Gamma_\varepsilon$ for some $\sigma_0, \varepsilon_0 > 0$, hence Γ_{ε_0} is an open set in Γ_ε and so in Γ_0.

It is

$$\text{(6.3)} \qquad \mathcal{C}_p(a) = \mathcal{C}_1(a) \cup \bigcup \mathcal{C}_{p,q}(a),$$

where $\mathcal{C}_{p,q}(a), q > p$ is not divisible by p, is the set of all the curves in $\mathcal{C}(a)$ that have the form (6.2) in some coordinate system at a, where $c(0) \neq 0$. In fact, if in (6.2) we have $v = \sum c_{p\nu} t^{p\nu}$ then the curve (6.2) is smooth (it suffices to change the parameter to $\tau = t^p$). In the remaining cases $v = a_p t^p + ... + a_{kp} t^{kp} + c(t)t^q$, where $c(0) \neq 0$ and $pk < q < p(k+1)$, and it suffices to replace the coordinates to $z_1' = z_1, v' = v - a_p z_1 - ... - a_{kp} z_1^k$ (it is a biholomorphism of $\mathbb{C}^n$ onto $\mathbb{C}^n$).

Let us notice that if a curve $\Gamma \ni a$ is smooth at a, then its proper inverse image $\bar{\Gamma}$ intersects S at a unique point: $\bar{\Gamma} \cap S = \{\bar{a}\}$ and in a transversall way.

Proposition 6. *Let Γ be a curve in $\mathcal{C}_{p,q}$, $p > 1$. Then its proper inverse image $\bar{\Gamma}$ is a curve and $\bar{\Gamma} \cap S = \{\bar{a}\}$; if $q > 2p$ then $\bar{\Gamma} \in \mathcal{C}_{p,q-p}(\bar{a})$, and if $q < 2p$ then $\bar{\Gamma} \in \mathcal{C}_{q-p}(\bar{a})$.*

Dalsza część książki dostępna w wersji pełnej.