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Agnieszka Rossa, Andrzej Szymański – University of Łódź, Faculty of Economics and Sociology
Department of Demography and Social Gerontology, 90-214 Łódź, 41/43 Rewolucji 1905 r. St.

Lesław Socha – Cardinal Stefan Wyszyński University in Warsaw
Faculty of Mathematics and Natural Sciences. School of Exact Sciences, Institute of Informatics
01-938 Warszawa, 1/3 Wóycickiego St.

TRANSLATOR AND PROOFREADING
Janusz Kwitecki

REVIEWER
Jan Paradysz

INITIATING EDITOR
Iwona Gos

TYPESETTING
Agnieszka Rossa

TECHNICAL EDITOR
Leonora Wojciechowska

COVER DESIGN
Katarzyna Turkowska

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Preface

Mortality is generally considered relatively easy to forecast, particularly when the forecasting horizon is short. In longer periods however, its course may be affected by various changes brought about by all kinds of disturbances and events. A case in point is the health crisis in Poland of the 1970s and 1980s [Okólski 2003]. In such cases, it is of key importance that appropriate assumptions and an adequate model are selected.

Mortality forecasting is usually supported by extrapolative models, making use of the regularity found in age patterns and trends of death rates or probabilities over time.

There are several reasons why one should learn more about mortality models. Forecasting of mortality has a wide range of applications outside the field of statistics and mathematics. It is of fundamental importance in such areas as funding of public or private pensions and life insurance. Annuity providers and policy makers use mortality projections to determine appropriate pension benefits, to assess retirement income or life insurance products, to hold additional reserving capital or to manage the long term demographic risk. Thus, one of the important question arises: What is the best way to forecast future mortality rates and to model the uncertainty of such forecasts? A key input to address this question is the development of advanced mortality modeling methodology.

These notes are an attempt to capture the stochastic nature of mortality by approaching the subject of mortality modeling and forecasting from a new theoretical point of view, using theory of stochastic differential equations, theory of fuzzy numbers and complex numbers.

The book is addressed to tertiary students, doctoral students and specialists in the fields of demography, life insurance, statistics and economics. This research project was funded by the National Science Center pursuant to its decision no. 2015/17/B/HS4/00927.

Abbreviation and notation

Throughout this book, the following abbreviations for mortality models have been adopted:

SLC	Standard Lee–Carter
LCH	Lee–Carter hybrid
DLC	Dynamic Lee–Carter
DDLC	Discrete Dynamic Lee–Carter model
LCH	Lee–Carter hybrid
DLCH	discrete Lee–Carter hybrid
FLC	Fuzzy Lee–Carter
EFLC	Extended Fuzzy Lee–Carter
MFLC	Modified Fuzzy Lee–Carter
CFLC	Complex-Function Lee–Carter
QVLC	Quaternion-Valued Lee–Carter
V	Vasiček
DV	Discrete Vasiček
VH	Vasiček hybrid
DVH	Discrete Vasiček hybrid
VHM	Vasiček hybrid moment
DVHM	Discrete Vasiček hybrid moment
MV	Modified Vasiček
DMV	Discrete Modified Vasiček
MVH	Modified Vasiček hybrid
DMVH	Discrete Modified Vasiček hybrid
MVHM	Modified Vasiček hybrid moment
DMVHM	Discrete Modified Vasiček hybrid moment
CIR	Cox–Ingersoll–Ross
DCIR	Discrete Cox–Ingersoll–Ross
CIRH	Cox–Ingersoll–Ross hybrid
DCIRH	Discrete Cox–Ingersoll–Ross hybrid
CIRHM	Cox–Ingersoll–Ross hybrid moment
DCIRHM	Discrete Cox–Ingersoll–Ross hybrid moment
MCIR	Modified Cox–Ingersoll–Ross
DMCIR	Discrete Modified Cox–Ingersoll–Ross
MCIRH	Modified Cox–Ingersoll–Ross hybrid
DMCIRH	Discrete Modified Cox–Ingersoll–Ross hybrid
MCIRHM	Modified Cox–Ingersoll–Ross hybrid moment
DMCIRHM	Discrete Modified Cox–Ingersoll–Ross hybrid moment
GOB	Giacometti–Ortobelli–Bertocchi
DGOB	Discrete Giacometti–Ortobelli–Bertocchi
GOBH	Giacometti–Ortobelli–Bertocchi hybrid
DGOBH	Discrete Giacometti–Ortobelli–Bertocchi hybrid
GOBHM	Giacometti–Ortobelli–Bertocchi hybrid moment
DGOBHM	Discrete Giacometti–Ortobelli–Bertocchi hybrid moment

MP	Milevsky–Promislow
DMP	Discrete Milevsky–Promislow
MMP	Modified Milevsky–Promislow
DMMP	Discrete Modified Milevsky–Promislow
DMPH	Discrete Milevsky–Promislow hybrid
MPHM	Milevsky–Promislow hybrid moment
DMPHM	Discrete Milevsky–Promislow hybrid moment
MMPH	Modified Milevsky–Promislow hybrid
DMMPH	Discrete Modified Milevsky–Promislow hybrid
MMPHM	Modified Milevsky–Promislow hybrid moment
DMMPHM	Discrete Modified Milevsky–Promislow hybrid moment
MP-2DF	Milevsky–Promislow, with 2 dependent filters
MPH-2DF	Milevsky–Promislow hybrid, with 2 dependent filters
MPHM-2DF	Milevsky–Promislow hybrid moment, with 2 dependent filters
DMPHM-2DF	Discrete Milevsky–Promislow hybrid moment with 2 dependent filters
MP-2IF	Milevsky–Promislow, with 2 independent filters
MPH-2IF	Milevsky–Promislow hybrid, with 2 independent filters
MPHM-2IF	Milevsky–Promislow hybrid moment, with 2 independent filters
DMPHM-2IF	Discrete Milevsky–Promislow hybrid moment with 2 independent filters
MP-VLF	Milevsky–Promislow with vector linear filter
MPH	Milevsky–Promislow hybrid
MPH-VLF	Milevsky–Promislow hybrid, with a vector linear filter
MPHM-VLF	Milevsky–Promislow hybrid moment, with a vector linear filter
DMPH-VLF	Discrete Milevsky–Promislow hybrid, with a vector linear filter

Introduction

The phenomenon of mortality has been studied for many centuries. In the early 3rd c., a Roman jurist, Domitius Ulpianus, created for fiscal purposes the so-called Ulpian table containing life expectancies for the citizens of the Roman Empire. As historical sources do not mention what calculation method and source materials he had used, the Ulpian table is mainly of historical value [Rosset 1979, pp. 102–103].

It is recognized that the father of the mortality table methodology is John Graunt (1620–1674), since his work [Graunt, 1662] where mortality of generations of London residents was examined. Graunt based his analysis on the records of London parishes, but did not specify which periods they concerned. Graunt's research was continued by an English astronomer Edmond Halley (1656–1742), who proposed mortality tables for the Wrocław population [Halley 1693].

The modern methodology for constructing mortality tables, also known as "life-tables", is credited to Chin L. Chiang (1914–2014) and his book [Chiang 1968]. The more works on life-tables and mortality models come from 19th c. [Gompertz 1825, Thiele, Sprague 1871], but it is only during the last decades that the mortality modeling methodology started to develop, as evidenced by numerous books on this subject [Rosset 1979, Keilman 1990, Okólski 1990, Benjamin, Pollard 1993, Kannisto 1994, Tabeau *et al.* 2001, Keilman 2005, Alho, Spencer 2005, Girosi, King 2006, Kędelski, Paradysz 2006, Rossa *et al.* 2011].

Since the introduction of the Lee–Carter model [Lee, Carter 1992] proposed to forecast the trend of age-specific mortality rates, a range of mortality models have been proposed with modeling the probability of death, the age-specific mortality rate or the force of mortality.

Among mortality models three main approaches can be identified: extrapolation, expectation and explanation [Pitacco 2004, Booth 2006, Tabeau *et al.* 2001]. The most common one is an extrapolative approach which uses a real or fuzzy variable functions of age and time to describe patterns and trends in death probabilities, mortality rates (or their transformations) and other measures [Heligman, Pollard 1980, Brouhns *et al.* 2002, Lee, Miller 2001, Renshaw, Haberman 2003a, 2003b, 2003c, 2006, 2008, Koissi, Shapiro 2006, Cairns *et al.* 2006, 2008a, 2008b, 2009, 2011, Denuit 2007, Debon *et al.* 2008, Haberman, Renshaw 2008, 2009, 2011, Hatzopoulos, Haberman 2011, Fung *et al.* 2017].

Mortality models can be divided also into two main categories: static and dynamic models. Models in the first group are based on some algebraic equations, while in dynamic models of the second group the force of mortality (the intensity process) is expressed as a solution of stochastic differential equations [Vasiček 1977, Cox *et al.* 1985a, 1985b, Janssen,Skiadas 1995, Milevsky, Promislow 2001, Dahl 2004, Biffis 2005, Biffis, Denuit 2006, Schrager 2006, Bravo, Braumann 2007, Yashin 2007, Hainaut, Devolder 2007, 2008, Luciano *et al.* 2008, Luciano, Vigna 2008, Plat 2009, Bayraktar *et al.* 2009, Biffis *et al.* 2010, Coelho *et al.* 2010, Giacometti *et al.* 2011, Russo*et al.* 2011, Wang*et al.* 2011, Hainaut 2012, Rossa, Socha 2013].

Unfortunately, the simple dynamic models based on stochastic differential equations can be inadequate to describe demographic processes. In particular, they may fail to explain evolution of the phenomena, meaning that their behavior changes in continuous time or discrete time intervals. To make up for this disadvantage, researchers put forward a new type of models, called hybrid models, which account for interactions between continuous and discrete dynamics.

Hybrid models, or switching models [Boukas 2005], are constructed as the generalizations of the models with switching points that have been already used for automatic control and for random structure models [Kazakov, Artemiev 1980] describing phenomena within mechanics, biology, economics or empirical sciences. The authors of some studies have proposed complex mortality models sharing characteristics with the hybrid models [Biffis, Denuit 2006, Biffis *et al.* 2010, Hainaut 2012, Rossa, Socha 2013].

For the purposes of this study, a hybrid system will henceforth be understood as a family of static or dynamic models where the switchings take place according to some switching rule. The dynamic models will be described using stochastic differential equations. There exists a class of equations for which analytical solutions of relatively complex structure can be found, therefore a new group of hybrid models will be proposed called the moment hybrid models. The idea underlying their construction involves the replacement of the stochastic models by equivalent differential equations for moments.

Another promising approach to mortality modeling offers theory of fuzzy numbers. It is well-known that the main difficulty in the applications of the Lee–Carter model is due to the assumed homogeneity of random terms. However, this property is not confirmed by the real-life data. The problem prompted search for solutions that could do without this assumption. One of the possible options is to set research in the framework of the fuzzy number theory. This line of thinking was adopted by [Koissi, Shapiro 2006], where empirical observations and parameters of the Lee–Carter model were converted into fuzzy symmetric triangular numbers.

Unfortunately, the Koissi–Shapiro model involves some difficulties, which arise from the necessity to find the minimum of a criterion function containing a max-type operator and cannot be solved using standard optimization algorithms. One approach to such a problem can be applying the Banach algebra of oriented fuzzy numbers (OFN) developed by [Kosiński *et al.* 2003]. The results of using this algebra to the Koissi–Shapiro model have been published in [Szymański, Rossa 2014].

A more sophisticated modification of the Koissi–Shapiro model involves the replacement of the Banach OFN algebra by the Banach C^*–algebra to allow the use of the Gelfand–Mazur theorem about isometric isomorphism between the C^*–algebra and the Banach algebra of complex functions and, consequently, to move the optimization problem into the framework of complex analysis. To our best knowledge, this is an innovative approach to mortality modeling.

This book has the following structure. In Chapter 1, basic mortality characteristics and some static and dynamic mortality models are discussed, especially the oldest historical mortality models (the so-called "mortality laws"), the well-known Lee–Carter model with its extensions and generalizations, the Vasiček and Cox–Ingersoll-Ross models, the Giacometti–Ortobelli–Bertocchi model and some variants of the Milevsky–Promislow model. Chapter 2 introduces theoretical backgrounds of hybrid modeling. In Chapter 3, hybrid counterparts of the dynamic models presented in Chapter 1 are provided and some estimation procedures are proposed. Chapter 4 discusses the theoretical underpinnings of the fuzzy mortality modeling based on the algebra of Oriented Fuzzy Numbers (OFN), whereas Chapter 5 presents mortality models from the perspective of the so-called modified fuzzy numbers (MFN) and complex functions. Chapter 6 illustrates results of estimations of some proposed models, the parameters of which were estimated using empirical mortality data sets. The comparative analysis of the models' prediction accuracy is also performed.

Chapter 1

Basic mortality characteristics and models

1.1. Introduction

Demographic models are an attempt to generalize and simplify real demographic processes by means of mathematical functions or a set of mathematical relations in order to approximate possible variations observed in the real data and to support demographic forecasting.

In this chapter basic notions, relations and some discrete-time as well as continuous-time extrapolative mortality models are introduced.

The main attention is focused on the well-known Lee–Carter model, its generalizations, the Vasiček and Cox–Ingersoll–Ross models as well as the Milevsky–Promislow and Giacometti–Ortobelli–Bertocchi models. They will be converted to hybrid models in Chapter 3.

1.2. Discrete-time mortality frameworks

1.2.1. Age-specific rates and probabilities of death

The definition of a mortality rate used in this book draws on the general definition of a cohort (or period) demographic rate defined as a ratio of the number of demographic events occurring in some defined cohort (or in a real population within some defined time period) to the time-to-exposure, understood as the number of time units lived by the cohort (or by the population during the given time period) [Preston *et al.* 2001, pp. 5–32].

If person-years are used in the denominator, a demographic rate is termed ”an annualized rate”. Below the definitions of both a cohort and a period annualized age-specific mortality rates are provided [Rossa *et al.* 2011, pp. 229–231].

An important notion used in the Definition 1.1 is "a cohort", defined as a real or hypothetical aggregate of individuals that experience a specific demographic event, e.g. births, during a specific time interval. The cohort is identified by the event itself and by its time frame.

For the purposes of this discussion, let index t indicate a calendar year from the given set $\{1, 2, \ldots, T\}$, and index x the attained age, meaning that it takes values from the set $\{0, 1, \ldots, X\}$, where X is the fixed upper age limit.

Definition 1.1. A cohort age-specific mortality rate $m_x^{(s)}$ in the s-th cohort is a ratio of the number of deaths, $D_x^{(s)}$, among individuals aged x years last birthday to the number of person-years, $K_x^{(s)}$, lived in the age range $[x, x+1)$

$$m_x^{(s)} = \frac{D_x^{(s)}}{K_x^{(s)}}. \tag{1.2.1}$$

Definition 1.2. A period age-specific mortality rate $m_{x,t}$ is a ratio of the number of deaths, $D_{x,t}$, among individuals in the age range $[x, x+1)$ years during the calendar year t to the number of person-years, $K_{x,t}$, lived in th age interval $[x, x+1)$ during this year

$$m_{x,t} = \frac{D_{x,t}}{K_{x,t}}. \tag{1.2.2}$$

It is worth noting that the denominators $K_x^{(s)}$ in (1.2.1) and $K_{x,t}$ in (1.2.2) can be treated as the number of individuals exposed to the risk of death in the given age interval or in the age-time interval, respectively. In the case of (1.2.2) the denominator is usually replaced by the midyear population $\bar{L}_{x,t}$, lived in the age range $[x, x+1)$ during the given year t. Therefore, period mortality rates (1.2.2) are often described as central death rates because of a midyear population used in the denominator.

For convenience (1.2.1), (1.2.2) are often expressed in thousands as

$$m_x^{(s)} = \frac{D_x^{(s)}}{K_x^{(s)}} \cdot 1\,000, \quad m_{x,t} = \frac{D_{x,t}}{K_{x,t}} \cdot 1\,000. \tag{1.2.3}$$

In a more general discrete approach, it is possible to consider an age interval $[x, x+n)$, where $n \in \mathbb{N}$ and $n > 1$. The cohort age-specific mortality rates (1.2.1) are then denoted as ${}_n m_x^{(s)}$ and the period age-specific mortality rates (1.2.2) as ${}_n m_{x,t}$.